Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-09-23

Problem 1. Here are some conceptual questions on the second derivative test. Let D be the Hessian determinant. When performing the second derivative test, there are a number of cases you consider (see last page of Lecture 08 slides):

- $D(a,b) > 0$ and $f_{xx}(a,b) > 0$
- $D(a,b) > 0$ and $f_{xx}(a,b) < 0$
- $D(a,b) < 0$

Regarding this procedure:

(a) What's the conclusion in each scenario?
(b) In the first two scenarios, why do we look at f_{xx} instead of f_{yy}? (Is there a difference?)
(c) The case $D(a,b) = 0$ is missing. Explain why it's missing.
(d) The case $D(a,b) > 0$ and $f_{xx}(a,b) = 0$ is also missing. Explain why it's missing.
(e) Can you find an example of a function $f(x,y)$ such that $(0,0)$ is a critical point of f, $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both strictly positive, but $(0,0)$ is *not* a local minimum?

(a) Local min in first, local max in second, saddle pt. in third.

(b) Assuming $D(a,b)>0$, examining

$$0 < D(a,b) + f_{xy}(a,b)^2 = f_{xx}(a,b) \cdot f_{yy}(a,b)$$

shows that f_{xx} and f_{yy} have the same sign (either both are positive or both are negative).

(c) It's missing b/c it's inconclusive. Consider x^4+y^4, x^4-y^4, and $-x^4-y^4$. $(0,0)$ is a critical pt. for all of these. All second order partial derivatives are zero (in particular $D=0$) but $(0,0)$ is a local min in the first, saddle in second, and local max in third.

(d) It's not possible: $0 < D(a,b) = \underbrace{f_{xx}(a,b)}_{=0} f_{yy}(a,b) - f_{xy}(a,b)^2 \leq 0$ results in a contradiction.

(e) $f(x,y) = x^2 + y^2 + 100xy$ works (check!)

Problem 2 (Stewart §14.7 #22). Consider the function $f(x, y) = x^2 y e^{-x^2-y^2}$. Show that

(a) Show that $(\pm 1, 1/\sqrt{2})$ are local maxima and $(\pm 1, -1/\sqrt{2})$ are local minima.

(b) **Actually f has infinitely many other critical points. Find them, and classify them.

(a) $\nabla f(x,y) = \langle -2x(x^2-1)ye^{-x^2-y^2}, x^2(1-2y^2)e^{-x^2-y^2} \rangle$

Note: $e^{-x^2-y^2} > 0$ (it is never $= 0$)

So $\nabla f(x,y) = 0$ amounts to

$$\begin{cases} x(x^2-1)y = 0 \\ x^2(1-2y^2) = 0. \end{cases}$$

Easy (but painful) to check the rest of (a) from here, so I'll omit it.

(b) The other solutions to the above eq. are when $x = 0$ and $y =$ anything.

$f(0, b) = 0$.